Approximation Numbers of Identity Operators between Symmetric Sequence Spaces

Aicke Hinrichs[1]

*Department of Mathematics, Friedrich-Schiller-University, Ernst-Abbe-Platz 1-4,
D-07743 Jena, Germany*
E-mail: nah@rz.uni-jena.de

Communicated by Allan Pinkus

We prove asymptotic formulas for the behavior of approximation quantities of identity operators between symmetric sequence spaces. These formulas extend recent results of Defant, Mastyło, and Michels for identities $l_p^n \hookrightarrow F_n$ with an n-dimensional symmetric normed space F_n with p-concavity conditions on F_n and $1 \leqslant p \leqslant 2$. We consider the general case of identities $E_n \hookrightarrow F_n$ with weak assumptions on the asymptotic behavior of the fundamental sequences of the n-dimensional symmetric spaces E_n and F_n. We give applications to Lorentz and Orlicz sequence spaces, again considerably generalizing results of Pietsch, Defant, Mastyło, and Michels. © 2002 Elsevier Science (USA)

Key Words: approximation numbers; identities; symmetric spaces.

1. INTRODUCTION

Of special interest for applications in approximation theory is the study of the asymptotic behavior of approximation quantities of finite dimensional identity operators. One of the first well-known examples is the computation of the approximation numbers of the identities id : $l_p^n \hookrightarrow l_q^n$ by Pietsch [6], see also [7, Chap. 11], for $1 \leqslant q \leqslant p \leqslant \infty$. In this case

$$a_k(\text{id} : l_p^n \hookrightarrow l_q^n) = (n - k + 1)^{1/q - 1/p}$$

for $k = 1, \ldots, n$.

This result was recently generalized by Defant, Mastyło and Michels in the case that $1 \leqslant p \leqslant 2$, see [1, 2]. If l_q^n is replaced by the nth section F_n of a p-concave symmetric Banach sequence space F, they obtain the asymptotic

formula

$$a_k(\text{id}: l_p^n \hookrightarrow F_n) \asymp \frac{\lambda_F(n-k+1)}{(n-k+1)^{1/p}},$$

where $\lambda_F(h) = ||\sum_{i=1}^h e_i||$ is the fundamental sequence associated with the Banach sequence space F with symmetric basis (e_i).

In [2] it is also conjectured that the formula

$$a_k(\text{id}: E_n \hookrightarrow F_n) \asymp \frac{\lambda_F(n-k+1)}{\lambda_E(n-k+1)} \qquad (1)$$

describes the asymptotic behavior of the identities for a much wider range of symmetric sequence spaces E and F, where F is naturally embedded into E. It is the purpose of this paper to confirm this conjecture if the fundamental sequences λ_E and λ_F satisfy certain weak regularity conditions. Applications to classical examples of Lorentz and Orlicz sequence spaces show that we obtain a far-reaching generalization of the corresponding results in [1, 2, 6].

We now describe the organization of the paper in some detail. In Section 2 we introduce some basic notation for approximation quantities and symmetric Banach sequence spaces. We state our main Theorem 1 which gives lower and upper estimates for approximation and Gelfand numbers in terms of the fundamental sequences of the involved symmetric sequence spaces. In Sections 3 and 4 we prove these estimates. The last Section 5 is concerned with applications to classical examples of symmetric sequence spaces. We consider Orlicz and Lorentz sequence spaces and compare our results to previously known results.

2. NOTATION AND PRELIMINARIES

We start with the introduction of some basic notation for approximation quantities of linear operators between Banach spaces and for symmetric sequence spaces. As usual, given Banach spaces E and F, the Banach space of all (bounded linear) operators from E into F equipped with the operator norm is denoted by $\mathscr{L}(E,F)$. For a (closed linear) subspace M of E, the inclusion mapping of M into E is denoted by J_M^E.

The approximation numbers of $T \in \mathscr{L}(E,F)$ are given by

$$a_k(T) = \inf\{||T - T_k|| : T_k \in \mathscr{L}(E,F) \text{ and rank } T_k < k\}$$

for $k = 1, 2 \ldots$. The Gelfand numbers of T are defined as

$$c_k(T) = \inf\{||TJ_M^E|| : M \subset E \text{ and codim } M < k\}.$$

For further information on approximation quantities of operators between Banach spaces, we refer the reader to [3, 8].

A basis $(e_k)_{k \geqslant 1}$ of a Banach space E is called symmetric if for all permutations π, all signs ε_k, and all sequences $(\xi_k)_{k \geqslant 1}$ such that $\sum_{k=1}^{\infty} \xi_k e_k \in E$, we have that $\sum_{k=1}^{\infty} \varepsilon_k \xi_{\pi(k)} e_k \in E$ and

$$\left\| \sum_{k=1}^{\infty} \varepsilon_k \xi_{\pi(k)} e_k \right\| = \left\| \sum_{k=1}^{\infty} \xi_k e_k \right\|.$$

By a symmetric Banach sequence space E we shall mean a Banach space of sequences of real numbers such that the unit vector basis e_k is a symmetric basis of E. The fundamental sequence of E is the sequence

$$\lambda_E(n) = \left\| \sum_{k=1}^{n} e_k \right\|.$$

For convenience, we set $\lambda_E(0) = 0$. With E_n we denote the span of $e_1, \ldots, e_n$ in E. Given symmetric Banach sequence spaces E, F, the formal identity from E_n to F_n is denoted by $\mathrm{id} : E_n \hookrightarrow F_n$.

The main objects under consideration in this note are approximation quantities for $\mathrm{id} : E_n \hookrightarrow F_n$. The following theorem gives sharp upper and lower bounds for Gelfand and approximation numbers of these identities if the fundamental sequences of the involved spaces satisfy certain combined regularity conditions.

We will demonstrate in Section 5 how this theorem extends and complements recent results on such approximation quantities in [1, 2].

THEOREM 1. *Let E, F be symmetric Banach sequence spaces such that*

1. *There exists $c_1 > 0$ such that*

$$\sum_{k=l+1}^{\infty} \frac{\lambda_E(k) - \lambda_E(k-1)}{\lambda_F(k)} \leqslant c_1 \frac{\lambda_E(l)}{\lambda_F(l)}$$

for $l = 1, 2, \ldots$.

2. *There exists $c_2 > 0$ such that*

$$\sum_{k=1}^{l} \frac{\lambda_F(k) - \lambda_F(k-1)}{\lambda_E(k)} \leqslant c_2 \frac{\lambda_F(l)}{\lambda_E(l)}$$

for $l = 1, 2, \ldots$.

Then

$$\frac{\lambda_F(n-k+1)}{(c_1+1)\lambda_E(n-k+1)} \leqslant c_k(\mathrm{id}: E_n \hookrightarrow F_n)$$

$$\leqslant a_k(\mathrm{id}: E_n \hookrightarrow F_n) \leqslant \frac{c_2\lambda_F(n-k+1)}{\lambda_E(n-k+1)}.$$

for $n = 1, 2, \ldots$ and $k = 1, \ldots, n$.

Observe that, up to equivalent renormings, it is no restriction to assume that $(\lambda_E(k) - \lambda_E(k-1))$ and $(\lambda_F(k) - \lambda_F(k-1))$ are nonincreasing, see [4, Proposition 3.a.7]. But this assumption is not needed for the theorem. On the other hand, it is satisfied in the considered concrete examples. We prove this theorem in the next two sections. It directly follows from Propositions 1 and 2 and the fact that $c_k(T) \leqslant a_k(T)$ for any operator T and all k.

Given two sequences α_n and β_n of positive real numbers we write $\alpha_n \prec \beta_n$ whenever there exists $c > 0$ such that $\alpha_n \leqslant c\beta_n$ for all n. Moreover, $\alpha_n \asymp \beta_n$ means that $\alpha_n \prec \beta_n$ and $\beta_n \prec \alpha_n$. We also use this notation for double sequences $(\alpha_{n,k})$ and $(\beta_{n,k})$ with the understanding that the involved constants depend neither on n nor on k.

3. BOUNDING THE APPROXIMATION NUMBERS

The following lemma is well-known. To keep the paper self-contained we include the proof here.

LEMMA 1. *Let E be a symmetric Banach sequence space. Let $\xi_1, \ldots, \xi_n$ be nonnegative reals such that $\xi_1 \geqslant \xi_2 \geqslant \cdots \geqslant \xi_n \geqslant 0$. Then*

$$\max_{1 \leqslant k \leqslant n} \lambda_E(k)\xi_k \leqslant \left\|\sum_{k=1}^n \xi_k e_k\right\| \leqslant \sum_{k=1}^n (\lambda_E(k) - \lambda_E(k-1))\xi_k.$$

Proof. The identity

$$\sum_{k=1}^n \xi_k e_k = \sum_{k=1}^{n-1} (\xi_k - \xi_{k+1}) \sum_{i=1}^k e_i + \xi_n \sum_{i=1}^n e_i$$

and the triangle inequality imply that

$$\left\|\sum_{k=1}^n \xi_k e_k\right\| \leqslant \sum_{k=1}^{n-1} (\xi_k - \xi_{k+1})\lambda_E(k) + \xi_n\lambda_E(n) = \sum_{k=1}^n (\lambda_E(k) - \lambda_E(k-1))\xi_k.$$

For the other inequality, let $S(h)$ be the symmetric group of all permutations of $\{1, \ldots, h\}$. Then the symmetry of the norm and the triangle inequality give that

$$\left\|\sum_{k=1}^{h} \xi_k e_k\right\| = \frac{1}{h!} \sum_{\pi \in S(h)} \left\|\sum_{k=1}^{h} \xi_{\pi(k)} e_k\right\| \geqslant \left\|\frac{1}{h!} \sum_{\pi \in S(h)} \sum_{k=1}^{h} \xi_{\pi(k)} e_k\right\|$$

$$= \left\|\sum_{k=1}^{h} \left(\frac{1}{h!} \sum_{\pi \in S(h)} \xi_{\pi(k)}\right) e_k\right\| = \left\|\sum_{k=1}^{h} \left(\frac{1}{h} \sum_{i=1}^{h} \xi_i\right) e_k\right\|$$

$$\geqslant \xi_h \lambda_E(h)$$

for $h = 1, \ldots, n$. Since $\|\sum_{k=1}^{h} \xi_k e_k\| \leqslant \|\sum_{k=1}^{n} \xi_k e_k\|$, we arrive at the claimed inequality. ∎

PROPOSITION 1. *Let E and F be symmetric Banach sequence spaces. Then*

$$a_k(\mathrm{id} : E_n \hookrightarrow F_n) \leqslant \sum_{i=1}^{n-k+1} \frac{\lambda_F(i) - \lambda_F(i-1)}{\lambda_E(i)}$$

for $n = 1, 2, \ldots$ and $k = 1, \ldots, n$.

Proof. For $h = 1, \ldots, n$, let $\mathrm{id}_h = \mathrm{id} : E_h \hookrightarrow F_h$. With slight abuse of notation, we denote by P_h the projection onto the span of $e_1, \ldots, e_h$ in E_n as well as in F_n. Observe that P_n is the identity on the span of $e_1, \ldots, e_n$. Then $\mathrm{rank}(P_n - P_{n-k+1})\mathrm{id}_n = k - 1$ implies that

$$a_k(\mathrm{id}_n) \leqslant \|\mathrm{id}_n - (P_n - P_{n-k+1})\mathrm{id}_n\| = \|P_{n-k+1}\mathrm{id}_n\|$$

$$= \|\mathrm{id}_{n-k+1} P_{n-k+1}\| \leqslant \|\mathrm{id}_{n-k+1}\|.$$

So it is enough to verify that for any $h = 1, 2, \ldots$ and all real numbers $\xi_1, \ldots, \xi_h$

$$\left\|\sum_{k=1}^{h} \xi_k e_k\right\|_{F_h} \leqslant \sum_{i=1}^{h} \frac{\lambda_F(i) - \lambda_F(i-1)}{\lambda_E(i)} \left\|\sum_{k=1}^{h} \xi_k e_k\right\|_{E_h}.$$

By the symmetry of both norms we may and do assume that $\xi_1 \geqslant \xi_2 \geqslant \cdots \geqslant \xi_n \geqslant 0$. Then the claimed inequality is an immediate consequence of Lemma 1. ∎

4. BOUNDING THE GELFAND NUMBERS

To bound the Gelfand numbers of identities between symmetric sequence spaces, we use the following well-known lemma about special vectors in subspaces of $\mathbb{R}^n$. For a proof see [7, 11.11.4]. We denote the cardinality of a set A with $|A|$.

LEMMA 2. *Let M be a subspace of $\mathbb{R}^n$ with codim $M < k$. Then there exists a vector $x = (\xi_1, \ldots, \xi_n) \in M$ such that $|\xi_k| \leqslant 1$ for $k = 1, \ldots, n$ and $|\{k : |\xi_k| = 1\}| \geqslant n - k + 1$.*

PROPOSITION 2. *Let E and F be symmetric Banach sequence spaces such that, for $n = 1, 2, \ldots$ and $l = 1, \ldots, n$,*

$$\sum_{i=l+1}^{n} \frac{\lambda_E(i) - \lambda_E(i-1)}{\lambda_F(i)} \leqslant c \frac{\lambda_E(l)}{\lambda_F(l)}. \tag{2}$$

Then

$$c_k(\mathrm{id} : E_n \hookrightarrow F_n) \geqslant \frac{\lambda_F(n-k+1)}{(c+1)\lambda_E(n-k+1)}$$

for $n = 1, 2, \ldots$ and $k = 1, \ldots, n$.

Proof. Let $l = n - k + 1$. Given a subspace $M \subset \mathbb{R}^n$ with codim $M < k$, we have to find a nontrivial $x \in M$ such that

$$\|x\|_{E_n} \leqslant (c+1) \frac{\lambda_E(l)}{\lambda_F(l)} \|x\|_{F_n}.$$

We show that we can take any $x \in M$ as given by Lemma 2. By symmetry of the involved norms, we may and do assume that

$$1 = \xi_1 = \cdots = \xi_l \geqslant \xi_{l+1} \geqslant \cdots \geqslant \xi_n \geqslant 0.$$

By Lemma 1 it is enough to show that for such x

$$\sum_{i=1}^{n} (\lambda_E(i) - \lambda_E(i-1))\xi_i \leqslant (c+1) \frac{\lambda_E(l)}{\lambda_F(l)} \max_{1 \leqslant i \leqslant n} \lambda_F(i)\xi_i.$$

Let $i_0 \geqslant l$ be such that

$$\lambda_F(i)\xi_i \leqslant \lambda_F(i_0)\xi_{i_0} \qquad \text{for } i = l, \ldots, n.$$

Since $1 = \xi_1 = \cdots = \xi_l$ we conclude from the monotonicity of λ_F that

$$\max_{1 \leqslant i \leqslant n} \lambda_F(i)\xi_i = \lambda_F(i_0)\xi_{i_0}. \tag{3}$$

So we are left to verify that

$$\sum_{i=1}^{n}(\lambda_E(i) - \lambda_E(i-1))\xi_i \leqslant (c+1)\frac{\lambda_E(l)}{\lambda_F(l)}\lambda_F(i_0)\xi_{i_0}. \tag{4}$$

It follows from $1 = \xi_1 = \cdots = \xi_l$ and (3) that

$$\sum_{i=1}^{l}(\lambda_E(i) - \lambda_E(i-1))\xi_i = \lambda_E(l) \leqslant \frac{\lambda_E(l)}{\lambda_F(l)}\lambda_F(i_0)\xi_{i_0}.$$

Moreover, it also follows from (3) and (2) that

$$\sum_{i=l+1}^{n}(\lambda_E(i) - \lambda_E(i-1))\xi_i \leqslant \sum_{i=l+1}^{n}\frac{\lambda_E(i) - \lambda_E(i-1)}{\lambda_F(i)}\lambda_F(i_0)\xi_{i_0}$$

$$\leqslant c\frac{\lambda_E(l)}{\lambda_F(l)}\lambda_F(i_0)\xi_{i_0}.$$

Adding up the last two inequalities proves (3) and the proposition. ∎

5. APPLICATION TO LORENTZ AND ORLICZ SPACES

We start this section with a modification of Theorem 1 in the case that

$$\lambda_E(k) \asymp k^{1/p} \quad \text{and} \quad \lambda_E(k) - \lambda_E(k-1) \asymp k^{1/p-1}$$

which in particular applies to the case that $E = l_p$ or, more generally, $E = l_{p,q}$ with $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$.

THEOREM 2. *Let $1 \leqslant q < p < \infty$. Let E and F be symmetric Banach sequence spaces such that*

(i) $\lambda_E(k) \asymp k^{1/p}$ *and* $\lambda_E(k) - \lambda_E(k-1) \asymp k^{1/p-1}$.

(ii) *There exists $c > 0$ such that $n^{1/q}\lambda_F(m) \leqslant c\lambda_F(mn)$ for $m, n = 1, 2, \ldots$.*
Then

$$a_k(\text{id} : E_n \hookrightarrow F_n) \asymp c_k(\text{id} : E_n \hookrightarrow F_n) \asymp \frac{\lambda_F(n-k+1)}{(n-k+1)^{1/p}}.$$

Proof.   We are going to show the inequalities

$$\sum_{k=l+1}^{\infty} \frac{\lambda_E(k) - \lambda_E(k-1)}{\lambda_F(k)} \prec \frac{l^{1/p}}{\lambda_F(l)}, \tag{5}$$

$$\sum_{k=1}^{l} \frac{\lambda_F(k) - \lambda_F(k-1)}{\lambda_E(k)} \prec \frac{\lambda_F(l)}{l^{1/p}}. \tag{6}$$

Application of Theorem 1 then finishes the proof.

To verify (5) and (6) we choose for given l an integer m such that $2^m \leqslant l < 2^{m+1}$. From assumption (i) we conclude that

$$\sum_{k=l+1}^{\infty} \frac{\lambda_E(k) - \lambda_E(k-1)}{\lambda_F(k)} \asymp \sum_{k=l+1}^{\infty} \frac{k^{1/p-1}}{\lambda_F(k)} \leqslant \sum_{k=2^m}^{\infty} \frac{k^{1/p-1}}{\lambda_F(k)} \asymp \sum_{k=m}^{\infty} \frac{2^{k/p}}{\lambda_F(2^k)}.$$

Assumption (ii) now implies that

$$2^{(k-m)/q} \lambda_F(2^m) \leqslant c \lambda_F(2^k) \qquad \text{for } k \geqslant m.$$

Using $1/q > 1/p$, we then arrive at

$$\sum_{k=m}^{\infty} \frac{2^{k/p}}{\lambda_F(2^k)} \prec \frac{2^{m/q}}{\lambda_F(2^m)} \sum_{k=m}^{\infty} 2^{k(1/p-1/q)} \asymp \frac{2^{m/p}}{\lambda_F(2^m)} \asymp \frac{l^{1/p}}{\lambda_F(l)}.$$

This proves (5).

Using (i) once again, we find that

$$\sum_{k=1}^{l} \frac{\lambda_F(k) - \lambda_F(k-1)}{\lambda_E(k)} \asymp \sum_{k=1}^{l} \lambda_F(k)(\lambda_E(k)^{-1} - \lambda_E(k+1)^{-1})$$

$$\asymp \sum_{k=1}^{l} \lambda_F(k) k^{-1/p-1} \asymp \sum_{k=1}^{m} 2^{-k/p} \lambda_F(2^k).$$

It follows from (ii) that

$$2^{(m-k)/q} \lambda_F(2^k) \leqslant c \lambda_F(2^m) \qquad \text{for } k = 1, \ldots, m.$$

Finally, using again $1/q > 1/p$, we get

$$\sum_{k=1}^{m} 2^{-k/p} \lambda_F(2^k) \asymp 2^{-m/q} \lambda_F(2^m) \sum_{k=1}^{m} 2^{k(1/q-1/p)} \asymp \frac{\lambda_F(2^m)}{2^{m/p}} \asymp \frac{\lambda_F(l)}{l^{1/p}},$$

which shows (6). ∎

To formulate our next corollary, we need the notion of a q-concave Banach lattice. A Banach lattice X is q-concave for some $1 \leqslant q < \infty$ if there exists c such that for all $x_1, \ldots, x_n \in X$

$$\left(\sum_{k=1}^{n} \|x_k\|^q \right)^{1/q} \leqslant c \left\| \left(\sum_{k=1}^{n} |x_k|^q \right)^{1/q} \right\|.$$

The smallest possible c is usually denoted by $M_q(X)$.

Trivially, a symmetric Banach sequence space is a Banach lattice. In this case, the operations of modulus and powers in the above inequality are simply applied coordinatewise. For more information on the concepts of p-convexity and q-concavity, we refer the reader to [5].

COROLLARY 1. *Let* $1 \leqslant q < p < \infty$. *Let* E *be a symmetric Banach sequence space with* $\lambda_E(k) \asymp k^{1/p}$ *and* $\lambda_E(k) - \lambda_E(k-1) \asymp k^{1/p-1}$. *Let* F *be a symmetric Banach sequence space such that there exists a q-concave Banach sequence space* G *with equivalent fundamental sequence, i.e.,* $\lambda_F(k) \asymp \lambda_G(k)$. *Then*

$$a_k(\mathrm{id} : E_n \hookrightarrow F_n) \asymp c_k(\mathrm{id} : E_n \hookrightarrow F_n) \asymp \frac{\lambda_F(n-k+1)}{(n-k+1)^{1/p}}.$$

Proof. This corollary is a direct consequence of Theorem 2 once we show that there exists $c > 0$ such that

$$n^{1/q} \lambda_G(m) \leqslant c \lambda_G(mn) \qquad \text{for } m, n = 1, 2, \ldots .$$

This inequality follows immediately from the definition of q-concavity using the vectors

$$x_k = \sum_{(k-1)m+1}^{km} e_i$$

for $k = 1, \ldots, n$. ∎

Now we give applications to special symmetric sequence spaces. For the definition and properties of the involved spaces we refer to [4].

EXAMPLE 1. *Lorentz sequence spaces* $l_{p,q}$. Let $1 < p_2 < p_1 < \infty$ and $1 \leqslant q_1, q_2 \leqslant \infty$. Then Corollary 1 applies with $p = p_1$, $q = p_2$, $E = l_{p_1,q_1}$, and $F = l_{p_2,q_2}$. Here we can choose $G = l_q = l_{p_2}$. So we have

$$a_k(\mathrm{id} : l_{p_1,q_1}^n \hookrightarrow l_{p_2,q_2}^n) \asymp c_k(\mathrm{id} : l_{p_1,q_1}^n \hookrightarrow l_{p_2,q_2}^n) \asymp (n - k + 1)^{1/p_2 - 1/p_1}.$$

EXAMPLE 2. *Lorentz sequence spaces* $d(w,r)$. Here $1 \leqslant r < \infty$ and $w = (w_n)$ is a nonincreasing null sequence of positive weights such that $w_1 = 1$ and such that the series $\sum w_n$ diverges. For simplicity, let us assume that $w = (w_n)$ is 1-regular, i.e., $\sum_{k=1}^n w_k \asymp nw_n$. Then

$$\lambda_{d(w,r)} = \left(\sum_{k=1}^n w_n \right)^{1/r} \asymp n^{1/r} w_n^{1/r}.$$

Assume now that $r < p$ and

$$w_m \leqslant cn^\alpha w_{mn}$$

for some c, some $\alpha < 1 - r/p$ and all m, n. Then we can choose $q \in (r, p)$ such that $w_m \leqslant cn^{1-r/q} w_{mn}$ which is easily seen to imply condition (ii) in Theorem 2 for $F = d(w,r)$. Thus we have

$$a_k(\mathrm{id} : l_p^n \hookrightarrow d_n(w,r)) \asymp c_k(\mathrm{id} : l_p^n \hookrightarrow d_n(w,r)) \asymp (n - k + 1)^{1/r - 1/p} w_{n-k+1}^{1/r}.$$

EXAMPLE 3. *Orlicz sequence spaces* l_φ. For an Orlicz sequence space $F = l_\varphi$, it is easily computed that

$$\lambda_F(n) = \frac{1}{\varphi^{-1}(1/n)}.$$

Then the assumption

$$\varphi^{-1}(st) \leqslant ct^{1/q} \varphi^{-1}(s)$$

for some c and all $s, t \in (0, 1]$ implies condition (ii) in Theorem 2. So, in this case we again have

$$a_k(\mathrm{id} : l_p^n \hookrightarrow l_\varphi^n) \asymp c_k(\mathrm{id} : l_p^n \hookrightarrow l_\varphi^n) \asymp \frac{1}{\varphi^{-1}(1/(n - k + 1))(n - k + 1)^{1/p}}.$$

Remark 1. Corollary 1 and the above examples show, that Theorem 2 has a wider range of application than Theorem 5.1 in [1]. Nevertheless, in some borderline cases where $E = l_p$ for $1 \leqslant p \leqslant 2$ and F is p-concave but not

q-concave for any $q < p$, that theorem is applicable but Theorem 2 is not. However, in the case that the only assumptions on the symmetric sequence spaces used are in terms of its fundamental sequences and not in terms of other geometric properties as e.g., q-convexity or type, our result seems to be optimal.

REFERENCES

1. A. Defant, M. Mastyło, and C. Michels, Summing inclusion maps between symmetric sequence spaces, *Trans. Amer. Math. Soc.*, to appear.
2. A. Defant, M. Mastyło, and C. Michels, Summing inclusion maps between symmetric sequence spaces, a survey, *in* ''Recent progress in functional analysis, Proceedings of the International Meeting on Functional Analysis held on the Occasion of the Seventieth Birthday of M. Valdivia in Valencia,'' July 3–7, 2000, pp. 43–60, 2001.
3. H. König, ''Eigenvalue Distribution of Compact Operators,'' Birkhäuser Verlag, Basel, 1986.
4. J. Lindenstrauss and L. Tzafriri, Classical Banach spaces—I, Springer-Verlag, Berlin, 1977.
5. J. Lindenstrauss, L. Tzafriri, Classical Banach spaces—II, Springer-Verlag, Berlin, 1979.
6. A. Pietsch, s-numbers of operators in Banach spaces, *Studia Math.* **51** (1974), 201–223.
7. A. Pietsch. ''Operator Ideals,'' VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.
8. A. Pietsch, ''Eigenvalues and s-Numbers,'' Cambridge Univ. Press, Cambridge, 1987.